Solution

Class 12 - Mathematics

2020-2021 - Paper-2

Section A

1. Here, u = tan x

   $\Rightarrow \frac{du}{dx}$ = $\sec^2 x$

   v = sec x $\Rightarrow \frac{dv}{dx}$ = sec x tan x

   $\therefore \frac{du}{dx} = \frac{\sec^2 x}{\sec x \cdot \tan x}$ or cosec x

2. Given:

   x = 2at, y = at^2

   $\frac{dx}{dt} = 2a; \frac{dy}{dt} = 2at$

   $\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\frac{\mathrm{dy}}{\mathrm{dt}}}{\frac{\mathrm{dx}}{\mathrm{dt}}}$ = t

   $\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{dx}^2} = \frac{\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\mathrm{dy}}{\mathrm{dx}}\right)}{\frac{\mathrm{dx}}{\mathrm{dt}}}$

   $= \frac{1}{2a}$

3. We have,

   $\frac{dy}{dx} = \frac{d}{dx}$ {e^x log (sin 2x)}

   = $e^x \frac{d}{dx}(\log(\sin 2x)\}$ + log (sin 2x)$\frac{d}{dx}(e^x)$

   = $e^x \cdot \left\{\frac{1}{\sin 2x} \cdot \cos 2x \cdot 2\right\}$ + log (sin 2x) e^x

   = $2e^x$ cot 2x + e^x log (sin 2x)

   = e^x{2cot 2x + log (sin 2x)}

4. Given, $f(x) = \begin{cases} \frac{\sin^{-1} x}{x}, & x \neq 0 \\ k & ,x = 0 \end{cases}$

   If f(x) is continuous at x = 0, then

   $\lim_{x \to 0} f(x) = f(0)$

   $\Rightarrow \lim_{x \to 0}\left(\frac{\sin^{-1} x}{x}\right) = f(0)$

   $\Rightarrow \lim_{x \to 0}\left(\frac{\sin^{-1} x}{x}\right) = k$

   $\Rightarrow$ k = 1 ... $\left[\because \lim_{x \to 0}\left(\frac{\sin^{-1} x}{x}\right) = 1\right]$

5. Let $y = \cos(\sin x)$

   $\therefore \frac{dy}{dx} = -\sin(\sin x)\frac{d}{dx}\sin x$

   $= -\sin(\sin x)\cos x$

   $= -\cos x.\sin(\sin x)$

6. The function f(x) = (w ○ v ○ u) (x) = sin (cos (x^2)) is a composition of the three functions u, v and w, where u(x) = x^2 , v(t) = cos t and w(s) = sin s. Put

   t = u(x) = x^2 and s = v(t) = cos t.

   Observe that $\frac{dw}{ds} = \cos s, \frac{ds}{dt}$ = - sin t and $\frac{dt}{dx}$ = 2x exist for all real x.

   Hence by a generalization of chain rule, we have

   $\frac{df}{dx} = \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx}$

   = (cos s) . (– sin t) . (2x)

   = – 2x sin x^2 . cos (cos x^2)

7. Since f(x) is continuous at x = -1

$\therefore \lim_{x \to -1} f(x) = f(-1)$

$\Rightarrow \lim_{x \to -1} \frac{x^2-2x-3}{x+1} = \lambda$ ...[$\because$ f(-1) = λ]

$\Rightarrow \lim_{x \to -1} \frac{(x-3)(x+1)}{x+1} = \lambda$

$\Rightarrow \lim_{x \to -1} (x-3) = \lambda$

$\Rightarrow -4 = \lambda$

So, f(x) is continuous at x = -1, if λ = -4

8. Here,

$y = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} \ldots .$ to ∞

$\frac{dy}{dx} = 0 - 1 + \frac{2x}{2!} - \frac{3x^2}{3!} - \frac{4x^3}{4!} + \cdots \infty$

$\frac{d^2y}{dx^2} = 0 - 0 + 1 - \frac{2x}{2!} + \frac{3x^2}{3!} - \frac{4x^3}{4!} + \cdots \infty$

$\frac{d^2y}{dx^2} = 1 - \frac{\mathrm{x}}{1!} + \frac{\mathrm{x}^2}{2!} - \frac{\mathrm{x}^3}{3!} + \cdots \infty$

$\frac{d^2y}{dx^2} = y$

9. f(x) = x^2e^{-x}

Differentiating w.r.t x, we get,

f'(x) = $-x^2e^{-x} + 2xe^{-x} = xe^{-x}(2-x)$

For increasing function, f'(x)≥ 0

$xe^{-x}(2-x) \geq 0$

$x(2-x) \geq 0$ [$\because$ e^{-x} is always positive]

$x(x-2) \leq 0$ [since - (x - 2) will change the inequality)

$Here\ x < 0\ \&\ (\ x-2) > 0\ \Rightarrow x\ < 0\ \&\ x > 2\ \Rightarrow 0 < x\ < 2$

$But\ when\ \ x > 0\ \&\ (\ x-2) < 0\ \Rightarrow x\ > 0\ \&\ x < 2$

$0 \leq x \leq\ 2$

10. Not that

f '(x) = $3x^2 - 6x + 4$

= $3(x^2 - 2x + 1) + 1$

= $3(x - 1)^2 + 1 > 0$, in every interval of R

Therefore, the function f is increasing on R.

11. Let x_1 , $x_2 \in$ R such that $x_1 > x_2$.

Therefore, $x_1 > x_2 \Rightarrow e^{x1} > e^{x2}$ ... [$\therefore$ e > 1 and $x_1 > x_2 \Rightarrow e^{x1} > e^{x2}$]

$\Rightarrow$ f(x_1) > f(x_2).

Thus, $x_1 > x_2$

$\Rightarrow$ f(x_2) > f(x_2) for all x_1 $x_2 \in$ R.

Therefore, f(x) is strictly increasing on R.

12. The given equation of the curve is

y = x^2 .... (i)

$\therefore$ Slope of tangent to (i) is

$\frac{dy}{dx} = 2x$ ..... (ii)

By the given condition,we have,

$\frac{dy}{dx} = x$ ..... (iii) [Slope = x-coordinate]

From (ii) and (iii)

2x = x

$\Rightarrow$ x = 0 & y = 0

Thus, the required point is (0, 0)

13. The given equation of curve is

y = $3x^2 + 4$ .... (i)

Slope $= m_1 = \frac{dy}{dx} = 6x$ .... (ii)
Now,
The given slope $m_2 = \frac{-1}{6}$
We are given that,
tangent to (i) is perpendicular to the tangent whose slope is $\frac{-1}{6}$
$\therefore m_1 \times m_2 = -1$
$\Rightarrow \quad 6x \times \frac{-1}{6} = -1$
$\Rightarrow$ x = 1
From (i)
y = 7
Therefore, the required point is (1, 7).

14. The given function is,
 $f(x) = 10^x$
 $\Rightarrow f'(x) = (10^x)(\log 10) > 0$ for all $x \in R$.
 Hence, $f(x) = 10^x$ is strictly increasing on R.

15. The equation of the curve is $y = x^2 - 5x + 6$. ...(i)
 Differentiating with respect to x, we get
 $\frac{dy}{dx} = 2x - 5$
 Now, m_1 =Slope of the tangent at (2, 0) $= \left(\frac{dy}{dx}\right)_{(2,0)} = 2 \times 2 - 5 = -1$
 and m_2 = Slope of the tangent at (3, 0) $= \left(\frac{dy}{dx}\right)_{(3,0)} = 2 \times 3 - 5 = 1$
 Clearly, $m_1 m_2 = -1 \times 1 = -1$
 Therefore,the tangents to two curves are perpendicular to each other.

16. Let the required point be $P(x_1, y_1)$. The given curve is
 $y = 2x^2 - 6x - 4$ .... (i)
 $\Rightarrow \quad \frac{dy}{dx} = 4x - 6 \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = 4x_1 - 6$
 Since the tangent at (x_1, y_1) is parallel to x-axis. Therefore,we have,
 $\left(\frac{dy}{dx}\right)_{(x_1,y_1)} = 0 \Rightarrow 4x_1 - 6 = 0 \Rightarrow x_1 = \frac{3}{2}$
 Since (x_1, y_1) lies on curve (i). Therefore,we have,
 $y_1 = 2x_1^2 - 6x_1 - 4$
 $\therefore x_1 = \frac{3}{2} \Rightarrow y_1 = 2(\frac{3}{2})^2 - 6\frac{3}{2} - 4 = -\frac{17}{2}$
 Therefore, the required point $\left(\frac{3}{2}, \frac{-17}{2}\right)$

Section B

17. i. (a) $4x + 4\sqrt{a^2 - x^2}$
 ii. (c) $\frac{dP}{dx} = 0$
 iii. (b) $\frac{a}{\sqrt{2}}$
 iv. (a) Square
 v. (d) $10\sqrt{2}$ cm

18. i. (a) $4(x^3 - 24x^2 + 144x)$
 ii. (a) Local maxima at $x = c_1$
 iii. (c) 4 cm
 iv. (b) 1024 cm^3
 v. (b) 0

Section C

19. Let y = cos (sin $\sqrt{ax+b}$) and u = ax + b

$\Rightarrow$ y = cos (sin $\sqrt{u}$)

Therefore, y = cos (sin v) where v = $\sqrt{u}$

let w = sin v

Therefore, y = cos w

Differentiating above equation w.r.t x,

$\therefore \frac{dy}{dx} = \frac{dy}{dw} \cdot \frac{dw}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx}$ ....By chain rule

$\therefore \frac{dy}{dx} = \frac{\mathrm{d}}{\mathrm{dw}}(\cos \mathrm{w}) \cdot \frac{\mathrm{d}}{\mathrm{dv}}(\sin \mathrm{v}) \cdot \frac{\mathrm{d}}{\mathrm{du}}(\sqrt{\mathrm{u}}) \cdot \frac{\mathrm{d}}{\mathrm{dx}}(\mathrm{ax}+\mathrm{b})$

= (-sin w) (cos v) $\left(\frac{1}{2\sqrt{\mathrm{u}}}\right) \cdot \left(\frac{\mathrm{d}}{\mathrm{dx}}(\mathrm{ax}) + \frac{\mathrm{d}}{\mathrm{dx}}(\mathrm{b})\right)$

= (-sin (sin v) (cos $\sqrt{u}$) $\left(\frac{1}{2\sqrt{\mathrm{ax+b}}}\right)$ (a + 0)

= (-sin (sin $\sqrt{u}$)) (cos $\sqrt{ax+b}$) $\left(\frac{1}{2\sqrt{\mathrm{ax+b}}}\right)$ (a)

= $\left(\frac{-a \cdot \cos \sqrt{ax+b}}{2\sqrt{ax+b}}\right)$ (sin (sin $\sqrt{ax+b}$))

20. The function f(x) = |x|

f(x) is not differentiable only at x = 0. This is because

$\lim\limits_{x \to 0^-} \frac{f(x)-f(0)}{x}$ = -1 and $\lim\limits_{x \to 0^+} \frac{f(x)-f(0)}{x}$ = 1

By extension, f(x) = |x − a| is not differentiable only at x = a. Therefore,

f(x) = |x − 3| + |x − 4| is not differentiable at x = 3

and

g(x) = |x − 3| + |x − 4| is not differentiable at x = 4.

21. We have,

$$f(x) = \begin{cases} \frac{|x-4|}{x-4}, & x \neq 4 \\ 0, & x = 4 \end{cases}$$

$$\Rightarrow f(x) = \begin{cases} \frac{-(x-4)}{x-4} = -1 & ; \quad x < 4 \\ \frac{x-4}{x-4} = 1 & ; \quad x > 4 \\ 0 & ; \quad x = 4 \end{cases} \text{..........} \left[\because |x-4| = \begin{cases} -(x-4), & x < 4 \\ x-4, & x \geq 4 \end{cases}\right]$$

When x < 4, we have f(x) = -1, which, being a constant function, is continuous at each point x < 4

Also, when x > 4, we have f(x) = 1, which being a constant function, is continuous at each point x > 4

Let us consider the point x = 4

We have,

$\lim\limits_{x \to 4^-} f(x) = \lim\limits_{x \to 4} -1 = -1$

$\lim\limits_{x \to 4^+} f(x) = \lim\limits_{x \to 4} 1 = 1$

and f(4) = 0

$\therefore \lim\limits_{x \to 4^-} f(x) \neq \lim\limits_{x \to 4^+} f(x)$

so, f(x) is not continuous at x = 4

Hence, f(x) is continuous everwhere, except at x = 4

22. Let the curves intersect at (x_1, y_1).

Therefore, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow \frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{b^2 x}{a^2 y}$

$\Rightarrow$ Slope of tangent at the point of intersection is $m_1 = \frac{b^2 x_1}{a^2 y_1}$

Again, $xy = c^2 \Rightarrow x\frac{dy}{dx} + y = 0$

$\Rightarrow \frac{dy}{dx} = \frac{-y}{x} \Rightarrow m_2 = \frac{-y_1}{x_1}$.

For orthoganality, $m_1 \times m_2 = -1$

$\Rightarrow \frac{b^2 x_1}{a^2 y_1} \times \frac{-y_1}{x_1} = -1$

$\Rightarrow \frac{b^2}{a^2} = 1 \; or \; a^2 - b^2 = 0.$

Which is the required solution.

OR

Given: f(x) = $(x - 5)^4$

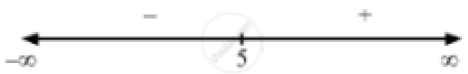
$\Rightarrow$ f'(x) = $4(x - 5)^3$

For a local maximum or a local minimum, we must have

f'(x) = 0

$\Rightarrow 4(x - 5)^3 = 0$

$\Rightarrow$ x = 5



Since f'(x) changes from negative to positive when x increases through 5, x = 5 as the point of local minima.

The local minimum value of the given function f(x) at x = 5 as given by $(5 - 5)^4 = 0$.

23. we have, f(x)=$kx^3 - 9x^2 + 9x + 3$

$\Rightarrow f'(x) = 3kx^2 - 18x + 9$

Since f(x) is increasing on R, therefore , $f'(x) > 0 \ \forall \ x \in R$

$\Rightarrow \ 3kx^2 - 18x + 9 > 0, \forall \ x \in \ R$

$\Rightarrow \ kx^2 - 6x + 3 > 0, \forall \ x \in \ R$

$\Rightarrow$ k>0 and 36-12k<0 $[\because \ ax^2 + bx + c > 0, \forall \ x \in \ R \Rightarrow \ a > 0 \ and \ discriminant < 0]$

$\Rightarrow \ k > 3$

Hence,f(x) is increasing on R, if k>3.

OR

Let $y = \sin\left(\tan^{-1} e^{-x}\right)$

$\therefore \frac{dy}{dx} = \cos\left(\tan^{-1} e^{-x}\right) \frac{d}{dx}\left(\tan^{-1} e^{-x}\right) \quad \left[\because \frac{d}{dx}\sin f(x) = \cos f(x) \frac{d}{dx} f(x)\right]$

$= \cos\left(\tan^{-1} e^{-x}\right) \frac{1}{1+\left(e^{-x}\right)^2} \frac{d}{dx} e^{-x} \quad \left[\because \frac{d}{dx}\tan^{-1} f(x) = \frac{1}{1+(f(x))^2} \frac{d}{dx} f(x)\right]$

$= \cos\left(\tan^{-1} e^{-x}\right) \frac{1}{1+e^{-2x}} e^{-x} \frac{d}{dx}(-x)$

$= -\frac{e^{-x}\cos\left(\tan^{-1} e^{-x}\right)}{1+e^{-2x}}$

24. Let $y = \frac{\cos x}{\log x}$

$\therefore \frac{dy}{dx} = \frac{\log x \frac{d}{dx}(\cos x) - \cos x \frac{d}{dx}(\log x)}{(\log x)^2}$ [By quotient rule]

$= \frac{\log x(-\sin x) - \cos x \cdot \frac{1}{x}}{(\log x)^2}$

$= \frac{-\left(\sin x \log x + \frac{\cos x}{x}\right)}{(\log x)^2}$

$= \frac{-(x \sin x \log x + \cos x)}{x(\log x)^2}$

OR

When x $\neq$ 0, then

$f(x) = \frac{\sin 3x}{x}$

We know that sin 3x as well as the identity function x are everywhere continuous. So, the quotient function $\frac{\sin 3x}{x}$ is continuous at each x $\neq$ 0

Let us consider the point x = 0

Given function is, $f(x) = \begin{cases} \frac{\sin 3x}{x} & , \text{ if } x \neq 0 \\ 4 & , \quad \text{if } x = 0 \end{cases}$

We have

(LHL at x = 0) = $\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0 + h) = \lim_{h \to 0} f(h)$

$\lim_{h \to 0} f(h) = \lim_{h \to 0}\left(\frac{\sin(3h)}{h}\right) = \lim_{h \to 0}\left(\frac{3\sin(3h)}{3h}\right) = 3$

(RHL at x = 0) = $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{h \to 0} f(0+h) = \lim\limits_{h \to 0} f(h)$

$= \lim\limits_{h \to 0}\left(\frac{\sin(3h)}{h}\right) = \lim\limits_{h \to 0}\left(\frac{3\sin(3h)}{3h}\right) = 3$

Also, f(0) = 4

$\therefore \lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^+} f(x) \neq f(0)$

Thus, f(x) is discontinuous at x = 0

Hence, the only point of discontinuity for f(x) is when x = 0

25. Let y = $e^{6x}\cos 3x$

$\therefore \frac{dy}{dx} = e^{6x}.\frac{d}{dx}\cos 3x + \cos 3x\frac{d}{dx}e^{6x}$

$= e^{6x}(-\sin 3x)\frac{d}{dx}(3x) + \cos 3x.e^{6x}\frac{d}{dx}(6x)$

$= -e^{6x}\sin 3x \times 3 + \cos 3x.e^{6x} \times 6$

= e^{6x}(-3 sin 3x + 6 cos 3x)

$\Rightarrow \frac{d^2y}{dx^2} = e^{6x}\frac{d}{dx}(-3\sin 3x + 6\cos 3x) + (-3\sin 3x + 6\cos 3x)\frac{d}{dx}e^{6x}$

$= e^{6x}(-3\cos 3x \times 3 - 6\sin 3x \times 3) + (-3\sin 3x + 6\cos 3x)e^{6x} \times 6$

= e^{6x}(-9 cos 3x - 18 sin 3x - 18 sin 3x + 36 cos 3x)

= e^{6x}(27cos 3x - 36sin3x)

= $9e^{6x}$(3 cos 3x - 4 sin 3x)

26. Here,

$f(x) = x\sqrt{1-x}, 0 < x < 1$

$\Rightarrow \mathrm{f}'(\mathrm{x}) = \sqrt{1-\mathrm{x}} + \mathrm{x}\frac{1}{2\sqrt{1-\mathrm{x}}}(-1)$

$= \sqrt{1-\mathrm{x}} - \frac{\mathrm{x}}{2\sqrt{1-\mathrm{x}}}$

$= \frac{2(1-\mathrm{x})-\mathrm{x}}{2\sqrt{1-\mathrm{x}}}$

f'(x) = 0

$\Rightarrow \frac{2-3x}{2\sqrt{1-x}} = 0$

$\Rightarrow$ 2 - 3x = 0

$\Rightarrow \mathrm{x} = \frac{2}{3}$

$f''(x) = \frac{1}{2}\left[\frac{\sqrt{1-x}(-3)-(2-3x)\left(\frac{-1}{2\sqrt{1-x}}\right)}{1-x}\right]$

$= \left[\frac{\sqrt{1-\mathrm{x}}(-3)+(2-3\mathrm{x})\left(\frac{-1}{2\sqrt{1-\mathrm{x}}}\right)}{2(1-\mathrm{x})}\right]$

$= \left[\frac{-6(1-\mathrm{x})+(2-3\mathrm{x})}{2(1-\mathrm{x})^{\frac{3}{2}}}\right]$

$= \frac{3x-4}{4(1-x)^{\frac{3}{2}}}$

Now,

$\mathrm{f}''\left(\frac{2}{3}\right) = -\frac{2}{4\left(\frac{1}{3}\right)^{\frac{3}{2}}}$ < 0,

Therefore, the second derivative test, x = $\frac{2}{3}$ is the point of local maxima and the local maximum value of 'f ' at x = $\frac{2}{3}$ is

f($\frac{2}{3}$) = $\frac{2}{3\sqrt{3}}$

27. Maximum value = 4, Minimum value = 2

We know that

$-1 \leq \sin\theta \leq 1$

$\therefore \ -1 \leq \sin 4x \leq 1$

Adding 3, on both sides, of above

We get

$-1+3 \leq \sin 4x+3 \leq 1+3$

$2 \leq |\sin 4x+3| \leq 4$

Hence min.Value is 2 and max value is 4.

28. The equation of a parabola is $y^2 = 4ax$, then,

On differentiating it with respect to x, we get

$\Rightarrow \frac{dy}{dx} = \frac{2a}{y}$

Then, the slope of the tangent at $(at^2, 2at)$ is

$\left.\frac{dy}{dx}\right]_{(at^2,2at)} = \frac{2a}{2at} = \frac{1}{t}$

Then, the equation of the tangent at $(at^2, 2at)$ is given by

$y-2at = \frac{1}{t}\left(x-at^2\right)$

$\Rightarrow ty - 2at^2 = x - at^2$

$\Rightarrow ty = x + at^2$

Now, then, slope of normal at $(at^2, 2at)$ is

$\frac{-1}{\text{Slope of the tangent at }\left(at^2,2at\right)} = -t$

Then, the equation of the normal at $(at^2, 2at)$ is given by:

$y - 2at = -t(x - at^2)$

$\Rightarrow y - 2at = -tx + at^3$

$\Rightarrow y = -tx + 2at + at^3$

Therefore, the equation of the normal at $(at^2, 2at)$ is $y = -tx + 2at + at^3$.

Section D

29. The given function is f(x) = $\begin{cases} x+5, \text{ if } x \leq 1 \\ x-5, \text{ if } x > 1 \end{cases}$

The function f is defined at all points of the real line.

Let k be the point on a real line.

Then, we have 3 cases i.e., k < 1, or k = 1 or k > 1

Now,

Case I: k < 1

Then, f(k) = k + 5

$\lim\limits_{x \rightarrow k} f(x) = \lim\limits_{x \rightarrow k}(x+5) = k+5 = f(k)$

Thus, $\lim\limits_{x \rightarrow k} f(x) = f(k)$

Hence, f is continuous at all real number less than 1.

Case II: k = 1

Then, f(k) = f(1) = 1 + 5 = 6

$\lim\limits_{x \rightarrow 1^-} f(x) = \lim\limits_{x \rightarrow 1^-}(x+5) = 1+5 = 6$

$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+}(x-5) = 1-5 = -4$

$\Rightarrow \lim\limits_{x \rightarrow k^-} \mathrm{f(x)} \neq \lim\limits_{\mathrm{x} \rightarrow \mathrm{k}^+} \mathrm{f(x)}$

Hence, f is not continuous at x = 1.

Case III: k > 1

Then, f(k) = k - 5

$\lim\limits_{x \rightarrow k} f(x) = \lim\limits_{x \rightarrow k}(x-5) = k-5$

Thus, $\lim\limits_{x \rightarrow k} f(x) = f(k)$

Hence, f is continuous at all real number greater than 1.

Therefore, x = 1 is the only point of discontinuity of f.

30. Let y = cosx cos2x cos3x ...... (i)

Taking log on both sides, we have

$\Rightarrow \log y = \log(\cos x \cos 2x \cos 3x)$

$\Rightarrow \log y = \log\cos x + \log\cos 2x + \log\cos 3x$
$\Rightarrow \frac{d}{dx}\log y = \frac{d}{dx}\log\cos x + \frac{d}{dx}\log\cos 2x + \frac{d}{dx}\log\cos 3x$
$\Rightarrow \frac{1}{y}.\frac{dy}{dx} = \frac{1}{\cos x}\frac{d}{dx}\cos x + \frac{1}{\cos 2x}\frac{d}{dx}\cos 2x + \frac{1}{\cos 3x}\frac{d}{dx}\cos 3x$
$\Rightarrow \frac{1}{y}.\frac{d}{dx} = \frac{1}{\cos x}(-\sin x) + \frac{1}{\cos 2x}(-\sin 2x)\frac{d}{dx}2x + \frac{1}{\cos 3x}(-\sin 3x)\frac{d}{dx}3x$
$\Rightarrow \frac{1}{y}.\frac{dy}{dx} = -\tan x - (\tan 2x)2 - \tan 3x(3)$
$\Rightarrow \frac{dy}{dx} = -y(\tan x + 2\tan 2x + 3\tan 3x)$
$\Rightarrow \frac{dy}{dx} = -\cos x\cos 2x.\cos 3x(\tan x + 2\tan 2x + 3\tan 3x)$

31. Given function is: f(x) = |x - 1| + |x + 1|
We have,
(LHL at x = -1) = $\lim\limits_{x\to -1^-} f(x) = \lim\limits_{h\to 0} f(-1-h)$
$= \lim\limits_{h\to 0}$ [|- 1 - h - 1| + |- 1 - h + 1|] = 2 + 0 = 0
(RHL at x = -1) = $\lim\limits_{x\to -1^+} f(x) = \lim\limits_{h\to 0} f(-1+h)$
$= \lim\limits_{h\to 0}$ [|- 1 + h - 1| + |- 1 + h + 1|] = 2 + 0 = 2
Also,
f(-1) = |- 1 - 1| + |- 1 + 1| = |-2| = 2
Now,
(LHL at x = 1) = $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{h\to 0} f(1-h)$
$= \lim\limits_{h\to 0}$ (|1 - h - 1| + |1 - h + 1|) = 0 + 2 = 2
(RHL at x = 1) = $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{h\to 0} f(1+h)$
$= \lim\limits_{h\to 0}$ (|1 + h - 1| + |1 + h + 1|) = 0 + 2 = 2
Also,
f(1) = |1 + 1| + |1 - 1| = 2
$\therefore \lim\limits_{x\to -1^-} f(x) = \lim\limits_{x\to -1^+} f(x) = f(-1)$ and $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^+} f(x) = f(1)$
Hence, f(x) is continuous at x = -1, 1

32. According to the question,if x = tan$\left(\frac{1}{a}\log y\right)$, then we have to show that $\left(1+x^2\right)\frac{d^2y}{dx^2} + (2x-a)\frac{dy}{dx} = 0$.
We shall use product rule of differentiation to prove the above result.
Now, x = tan$\left(\frac{1}{a}\log y\right)$
$\Rightarrow \tan^{-1} x = \frac{1}{a}\log y$
$\Rightarrow$ a tan ^{-1}x = log y
On differentiating both sides w.r.t x, we get,
$a \times \frac{1}{1+x^2} = \frac{1}{y}\cdot\frac{dy}{dx}$
$\Rightarrow \quad \left(1+x^2\right)\frac{dy}{dx} = ay$
Again, differentiating both sides w.r.t x, we get,
$\left(1+x^2\right)\cdot\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx}\cdot\frac{d}{dx}\left(1+x^2\right) = \frac{d}{dx}(ay)$[By using product rule of derivative]
$\Rightarrow \quad \left(1+x^2\right)\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot(2x) = a\cdot\frac{dy}{dx}$
$\Rightarrow \quad \left(1+x^2\right)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - a\frac{dy}{dx} = 0$
$\therefore \quad \left(1+x^2\right)\frac{d^2y}{dx^2} + (2x-a)\frac{dy}{dx} = 0$
Hence Proved.

OR

Left hand limit= $\lim\limits_{x\to\pi^-} f(x) = \lim\limits_{x\to\pi^-}(Kx+1)$
$= \lim\limits_{h\to 0}[K(\pi-h)+1]$
$= K\pi + 1$

Right hand limit = $\lim\limits_{x \to \pi^+} f(x) = \lim\limits_{x \to \pi^+} \cos x$

$= \lim\limits_{h \to 0} \cos(\pi + h) = \lim\limits_{h \to 0} - \cos h$

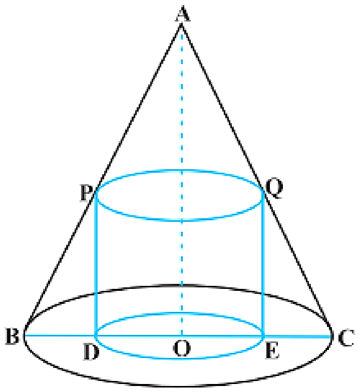
= -cos 0 = - 1

Therefore,

$K\pi + 1 = -1$

$\Rightarrow K = \frac{-2}{\pi}$

33. Let OC = r be the radius of the cone and OA = h be its height. Let a cylinder with radius OE = x inscribed in the given cone



The height QE of the cylinder is given by $\frac{\text{QE}}{\text{OA}} = \frac{\text{EC}}{\text{OC}}$ (since ΔQEC $\sim$ ΔAOC)

or $\frac{\text{QE}}{h} = \frac{r-x}{r}$

or QE = $\frac{h(r-x)}{r}$

Let S be the curved surface area of the given cylinder. Then

S ≡ S(x) = $\frac{2\pi x h(r-x)}{r} = \frac{2\pi h}{r}\left(rx - x^2\right)$

$\Rightarrow \quad S'(x) = \frac{2\pi h}{r}(r - 2x)$

$\Rightarrow \quad S''(x) = \frac{-4\pi h}{r}$

Now S' (x) = 0 gives x = $\frac{r}{2}$.

Since S"(x) < 0 for all x, S"$\left(\frac{r}{2}\right)$ < 0.

So, x = $\frac{r}{2}$ is a point of maxima of S. Hence, the radius of the cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone.

34. We have, $f\,(x) = \tan^{-1}(\sin x + \cos x)$

$\therefore f'\,(x) = \frac{1}{1+(\sin x+\cos x)^2}.(\cos x - \sin x)$

$= \frac{1}{1+\sin^2 x+\cos^2 x+2\sin x.\cos x}(\cos x - \sin x)$

$= \frac{1}{(2+\sin 2x)}(\cos x - \sin x)$

$\left[\because \sin 2x = 2\sin x \cos x \ and \ \sin^2 x + \cos^2 x = 1\right]$

For $f'\,(x) \geqslant 0$

$\frac{1}{(2+\sin 2x)}.(\cos x - \sin x) \geqslant 0$

$\Rightarrow \cos x - \sin x \geqslant 0 \ \left[\because (2 + \sin 2x) \geqslant 0 \ in \ \left(0, \frac{\pi}{4}\right)\right]$

$\Rightarrow \cos x \geqslant \sin x$

Which is true, if $x \in \left(0, \frac{\pi}{4}\right)$

Hence, f(x) is an increasing function in $\left(0, \frac{\pi}{4}\right)$.

35. Here, it is given the function

f(x) = 2cos x + x, where 0 < x < π.

$\Rightarrow$ f'(x) = - 2 sin x + 1

At points of local maximum and minimum, we must have

f'(x) = 0

$\Rightarrow$ -2 sin x + 1 = 0 $\Rightarrow$ sin x = $\frac{1}{2} \Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$

Therefore, $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$ are possible points of local maximum or minimum.

Now, we test the function at these points.
Clearly, f''(x) = - 2 cos x
At x = $\frac{\pi}{6}$: we have,
$f''\left(\frac{\pi}{6}\right)=-2\cos\frac{\pi}{2}=-\sqrt{3}<0$
Thus, $x=\frac{\pi}{6}$ is a point of local maximum. The local maximum value of f(x) is
$f\left(\frac{\pi}{6}\right)=2\cos\frac{\pi}{6}+\frac{\pi}{6}=\sqrt{3}+\frac{\pi}{6}$
At x = $\frac{5\pi}{6}$: We have,
$f''\left(\frac{5\pi}{6}\right)=-2\cos\frac{5\pi}{6}=\sqrt{3}>0$
Therefore, x= $\frac{5\pi}{6}$ is a point of local minimum.
The local minimum value of f(x) is $f\left(\frac{5\pi}{6}\right)=2\cos\frac{5\pi}{6}+\frac{5\pi}{6}=-\sqrt{3}+\frac{5\pi}{6}$.

OR

Given f(x) = $\sin^2$ x – cos x, x $\in$ [0, π]
$\Rightarrow$ f'(x) = 2sinx cosx + sinx = sinx(2cosx + 1)
For critical points, f'(x) =0 ==> sinx (2cosx + 1) = 0
$\Rightarrow$ sinx = 0, 1 + 2cosx = 0
$\Rightarrow$ x = 0, $\frac{2\pi}{3}$, $\pi \in$[0, π]
Now, f(0) = $\sin^2$ 0 - Cos 0 = -1, f$\left(\frac{2\pi}{3}\right)$ = $\sin^2\frac{2\pi}{3}-\cos\frac{2\pi}{3}$ = $\frac{3}{4}+\frac{1}{2}=\frac{5}{4}$, f($\pi$) = $\sin^2\pi$ - cos π =1
So, absolute maximum value = 5/4 and absolute minimum value = -1.

Section E

36. Let $\mathbf{u}=\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$
Put x = sinθ
$\Rightarrow \theta$ = $\sin^{-1}$x
$\Rightarrow u=\tan^{-1}\left(\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\right)$
$\Rightarrow u=\tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right)$
$\Rightarrow$ u = $\tan^{-1}$(tan θ) ...(i)
And
Let $v=\sin^{-1}(2x\sqrt{1-x^2})$
$v=\sin^{-1}(2\sin\theta\sqrt{1-\sin^2\theta})$
v = $\sin^{-1}$(2 sin θ cos θ)
v = $\sin^{-1}$(sin 2θ) ...(ii)
Here,
$-\frac{1}{\sqrt{2}}<x<\frac{1}{\sqrt{2}}$
$\Rightarrow -\frac{1}{\sqrt{2}}<\sin\theta<\frac{1}{\sqrt{2}}$
$\Rightarrow -\frac{\pi}{4}<\theta<\frac{\pi}{4}$
So, from equation (i)
$u=\theta\left[\text{ since, }\tan^{-1}(\tan\theta)=\theta,\text{ if }\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\right]$
$\Rightarrow$ u = $\sin^{-1}$x
Differentiating it with respect to x,
$\frac{du}{dx}=\frac{1}{\sqrt{1-x^2}}$ ....(iii)
from equation (ii),
$v=2\theta\left[\text{ since, }\sin^{-1}(\sin\theta)=\theta,\text{ if }\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\right]$
$\Rightarrow$ v = 2$\sin^{-1}$x
Differentiating it with respect to x,
$\frac{dv}{dx}=\frac{2}{\sqrt{1-x^2}}$ ...(iv)
Dividing equation (iii) by (iv),

$\frac{\frac{du}{dx}}{\frac{dv}{dx}} = \left(\frac{1}{\sqrt{1-x^2}}\right)\left(\frac{\sqrt{1-x^2}}{2}\right)$

$\therefore \frac{du}{dv} = \frac{1}{2}.$

OR

$x\sqrt{1+y} = -y\sqrt{1+x}$

Squaring both sides

$x^2(1+y) = y^2(1+x)$

$x^2 + x^2y = y^2 + xy^2$

$x^2 - y^2 + x^2y - xy^2 = 0$

$(x-y)(x+y) + xy(x-y) = 0$

$(x-y)[x+y+xy] = 0$

$x+y+xy = 0$

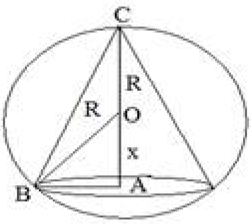
$y(1+x) = -x$

$y = \frac{-x}{1+x}$

$\frac{dy}{dx} = -\left[\frac{(1+x)(1)-(x)(1)}{(1+x)^2}\right]$

$= -\left[\frac{1+x-x}{(1+x)^2}\right]$

$\frac{dy}{dx} = \frac{-1}{(1+x)^2}$

37. 

$v = \frac{1}{3}\pi r^2 h \left[r^2 = \sqrt{R^2 - x^2}\right]$

$V = \frac{1}{2}\pi.\left(R^2 - x^2\right).(R+x)$

$\frac{dy}{dx} = \frac{1}{3}\pi\left[\left(R^2 - x^2\right)(1) + (R+x)(-2x)\right]$

$= \frac{1}{3}\pi\left[(R+x)(R-x) - 2x(R+x)\right]$

$= \frac{1}{3}\pi(R+x)[R-x-2x]$

$= \frac{1}{3}\pi(R+x)(R-3x)$....(1)

Put $\frac{dv}{dr} = 0$

R = - x (neglecting)

R = 3x

$\frac{R}{3} = x$

On again differentiating equation (1)

$\frac{d^2v}{dx^2} = \frac{1}{3}\pi\left[(R+x)(-3) + (R-3x)(1)\right]$

$= \left.\frac{d^2v}{dx^2}\right]_{x=\frac{R}{3}} = \frac{1}{3}\pi\left[\left(R+\frac{R}{3}\right)(-3) + \left(R - 3.\frac{R}{3}\right)\right]$

$\frac{1}{3}\pi\left[\frac{4R}{3} \times -3 + 0\right]$

$= \frac{-1}{3}\pi 4R$

$\frac{d^2v}{dx^2} < 0$ Hence maximum

Now $v = \frac{1}{3}\pi\left[\left(R^2 - x^2\right)(R+x)\right]\left[x = \frac{R}{3}\right]$

$v = \frac{1}{3}\pi\left[\left(R^2 - \left(\frac{R}{3}\right)^2\right)\left(R + \left(\frac{R}{3}\right)\right)\right]$

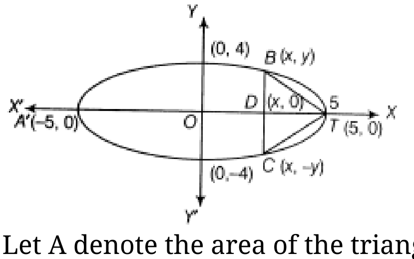
$= \frac{1}{3}\pi\left[\frac{8R^2}{9} \times \frac{4R}{3}\right]$

$v=\frac{8}{27}\left(\frac{4}{3}\right)\pi R^{3}$
$v=\frac{8}{27}$ Volume of sphere
Volume of cone $=\frac{8}{27}$ of volume of sphere.

OR

Let $y=(x\cos x)^{x}+(x\sin x)^{\frac{1}{x}}$
Putting u = (x cos x)^{x} and $v=(x\sin x)^{\frac{1}{x}}$
we have y = u + v
$\therefore \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}$ ....(i)
Now u = (x cos x)^{x}
$\Rightarrow \log u=\log(x\cos x)^{x}=x\log(x\cos x)$
$\Rightarrow \log u=x(\log x+\log\cos x)$
$\Rightarrow \frac{d}{dx}\log u=\frac{d}{dx}\{x(\log x+\log\cos x)\}$
$\Rightarrow \frac{1}{u}\frac{du}{dx}=x\left[\frac{1}{x}+\frac{1}{\cos x}.(-\sin x)\right]+(\log x+\log\cos x).1$
$\Rightarrow \frac{1}{u}\frac{du}{dx}=[1-x\tan x+\log(x\cos x)]$
$\Rightarrow \frac{du}{dx}=u[1-x\tan x+\log(x\cos x)]$
$\Rightarrow \frac{du}{dx}=(x\cos x)^{x}[1-x\tan x+\log(x\cos x)]$ .....(ii)
Again $v=(x\sin x)^{\frac{1}{x}}$
$\Rightarrow \log v=\log(x\sin x)^{\frac{1}{x}}=\frac{1}{x}\log(x\sin x)$
$\Rightarrow \log v=\frac{1}{x}(\log x+\log\sin x)$
$\Rightarrow \frac{d}{dx}\log v=\frac{d}{dx}\left\{\frac{1}{x}(\log x+\log\sin x)\right\}$
$\Rightarrow \frac{1}{v}\frac{dv}{dx}=\frac{1}{x}\left[\frac{1}{x}+\frac{1}{sinx}.\cos x\right]+(\log x+\log\sin x)\left(\frac{-1}{x^{2}}\right)$
$\Rightarrow \frac{1}{v}\frac{dv}{dx}=\left[\frac{1}{x^{2}}+\frac{\cot x}{x}-\frac{\log(x\sin x)}{x^{2}}\right]$
$\Rightarrow \frac{dv}{dx}=v\left[\frac{1}{x^{2}}+\frac{\cot x}{x}-\frac{\log(x\sin x)}{x^{2}}\right]$
$\Rightarrow \frac{dv}{dx}=(x\sin x)^{\frac{1}{x}}\left[\frac{1}{x^{2}}+\frac{\cot x}{x}-\frac{\log(x\sin x)}{x^{2}}\right]$ ...(iii)
Putting the values from eq. (ii) and (iii) in eq. (i)
$\frac{d}{dx}=(x\cos x)^{x}[1-x\tan x+\log(x\cos x)]+(x\sin x)^{\frac{1}{x}}\left[\frac{1}{x^{2}}+\frac{\cot x}{x}-\frac{\log(x\sin x)}{x^{2}}\right]$

38. Given equation of ellipse is
$\frac{x^{2}}{25}+\frac{y^{2}}{16}=1.$
Here, a = 5, b = 4
$\therefore$ a > b
So, major axis is along X-axis.
Let $\triangle$BTC be the isosceles triangle which is inscribed in the ellipse and OD = x, BC = 2y and TD = 5 - x.



Let A denote the area of the triangle. Then, we have
$A=\frac{1}{2}\times$ base $\times$ height $=\frac{1}{2}\times BC\times TD$
$\Rightarrow \quad A=\frac{1}{2}2y(5-x)\Rightarrow A=y(5-x)$
Therefore, on squaring both sides, we get,
$A^{2}=y^{2}(5-x)^{2}\ldots\ldots\ldots(i)$
Now, $\frac{x^{2}}{25}+\frac{y^{2}}{16}=1$
$\Rightarrow \quad \frac{y^{2}}{16}=1-\frac{x^{2}}{25}$

$\Rightarrow \quad y^2 = \frac{16}{25}\left(25 - x^2\right)$

On putting value of y^2 in Eq. (i), we get

$A^2 = \frac{16}{25}\left(25 - x^2\right)(5 - x)^2$

Let $A^2 = Z$

Then, $Z = \frac{16}{25}\left(25 - x^2\right)(5 - x)^2$

Therefore,on differentiating both sides w.r.t x, we get,

$\frac{dZ}{dx} = \frac{16}{25}\left[\left(25 - x^2\right)2(5 - x)(-1) + (5 - x)^2(-2x)\right]$ [by using product rule of derivative]

$= \frac{16}{25}(-2)(5 - x)^2(2x + 5)$

$= \frac{-32}{25}(5 - x)^2(2x + 5)$

For maxima or minima, put $\frac{dZ}{dx} = 0$

$\Rightarrow -\frac{32}{25}(5 - x)^2(2x + 5) = 0 \Rightarrow x = 5, -\frac{5}{2}$

Now, when x = 5, then

$Z = \frac{16}{25}(25 - 25)(5 - 5)^2 = 0$

Which is not possible.

So, x = 5 is rejected.

$\therefore \quad x = -\frac{5}{2}$

Now, $\frac{d^2Z}{dx^2} = \frac{d}{dx}\left[-\frac{32}{25}(5 - x)^2(2x + 5)\right]$

$= \frac{32}{25}\left[(5 - x)^2 \cdot 2 - (2x + 5) \cdot 2(5 - x)\right]$

$= -\frac{64}{25}(5 - x)(-3x) = \frac{192x}{25}(5 - x)$

$\therefore$ At $x = \frac{-5}{2}, \left(\frac{d^2Z}{dx^2}\right)_{x=-\frac{5}{2}} < 0$

$\Rightarrow$ Z is maximum.

$\therefore$ Area A is maximum, when $x = -\frac{5}{2}$ and y = 12

Clearly,

$Z = A^2 = \frac{16}{25}\left(25 - \frac{25}{4}\right)\left[5 + \frac{5}{2}\right]^2$

$= \frac{16}{25} \times \frac{75}{4} \times \frac{225}{4} = 3 \times 225$

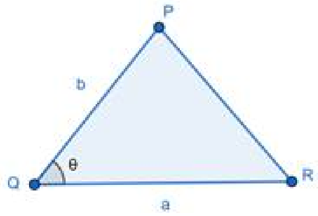
$\therefore$ The maximum area, $A = \sqrt{3 \times 225} = 15\sqrt{3}$ sq units.

OR

Given, The length two sides of a triangle are ‘a’ and ‘b’

Angle between the sides ‘a’ and ‘b’ is θ.

Then,the area of the triangle is maximum.



Let us consider,

The area of the $\triangle$PQR is given be

$A = \frac{1}{2}ab\sin\theta$ ... (i)

For finding the maximum/ minimum of given function, we can find it by differentiating it with θ and then equating it to zero. This is because if the function A (θ) has a maximum/minimum at a point c then A’(c) = 0.

Differentiating the equation (i) with respect to θ,we get

$\frac{dA}{d\theta} = \frac{d}{d\theta}\left[\frac{1}{2}ab\sin\theta\right]$

$\frac{dA}{d\theta} = \frac{1}{2}ab\cos\theta$ ... (ii)

[Since $\frac{d}{dx}(\sin\theta) = \cos\theta$]

To find the critical point, we need to equate equation (ii) to zero.we get

$\frac{dA}{d\theta} = \frac{1}{2}ab\cos\theta = 0$

Cos θ = 0

$\theta = \frac{\pi}{2}$

Now to check if this critical point will determine the maximum area, we need to check with second differential which needs to be negative.

Consider differentiating the equation (ii) with θ,we get

$\frac{d^2A}{d\theta^2} = \frac{d}{d\theta}\left[\frac{1}{2}ab\cos\theta\right]$

$\frac{d^2A}{d\theta^2} = -\frac{1}{2}ab\sin\theta$ ... (ii)

[Since $\frac{d}{dx}(\cos\theta) = -\sin\theta$]

Now let us find the value of

$\frac{d^2A}{d\theta^2}{}_{\theta=\frac{\pi}{2}} = -\frac{1}{2}\text{ ab }\sin\left(\frac{\pi}{2}\right) = -\frac{1}{2}ab$

As $\frac{d^2A}{d\theta^2}{}_{\theta=\frac{\pi}{2}} = -\frac{1}{2}ab < 0$, therefore, function A is maximum at $\theta = \frac{\pi}{2}$

Therefore, the area of the triangle is maximum when $\theta = \frac{\pi}{2}$